Identification of the orbit semi-major axis using frequency properties of onboard magnetic field measurements

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\textbf{ABSTRACT}

In this paper a novel algorithm for estimating the satellite's semi-major axis is presented. The procedure makes solely use of measurements of the magnitude of the magnetic field and it is therefore independent of the satellite's attitude. Relying on the fact that the orbital motion is nearly periodic in nature and that the Earth's magnetic field has a spherical harmonic behavior, the magnetic signal measured on board is analyzed in the frequency domain. The identification procedure calculates characteristic frequencies of the satellite motion from the magnetic spectrum and relates them to the orbit semi-major axis. Simulations, hardware-in-the loop and real data analyses have been performed. They prove that the algorithm is capable of estimating the semi-major axis well within 1 km for a broad range of orbits. Considering the tiny requirements in terms of measurement and computational burdens, the procedure looks appealing especially for very small satellites.

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1. Introduction

Autonomous onboard determination of the satellite’s orbit is a challenging task for spacecraft of a very small size. Nowadays, specialized space-rated GPS receivers may yield extraordinarily accurate onboard estimates of the satellite’s position and velocity in Low Earth Orbit (LEO). Moreover, GPS technology provides a compact equipment suited to any satellite of medium and large size. However, although in the last decade very small and light space GPS receivers have been developed \cite{1} and successfully applied to nanosatellites \cite{2}, tight requirements in terms of power and mass combined with economical factors make this solution generally problematic for satellites with mass of a few kilograms or less. Thus, some researches have focused on the problem of onboard identification of a spacecraft’s position using small and power efficient sensors. Most of these works proposed the utilization of the Kalman filter for correcting model predictions with onboard measurements of the magnetic field \cite{3–6} complemented with measurements from Sun sensor \cite{7}, horizon sensor \cite{8} or gyro \cite{9,10}. All cited works yield estimates of the actual position of a spacecraft by operating in the time-domain.

The intuition behind this study is that the spherical harmonic behavior of the Earth's magnetic field and the periodic motion of the orbiting body promote the analysis of onboard magnetic measurements in the frequency-domain for gaining information about the orbital motion of a satellite. Thus, this paper illustrates a novel method that estimates the semi-major axis of the orbit of a spacecraft based only on the spectral analysis of magnetic measurements. The proposed procedure is characterized by a small computational burden because it does not require any evaluation of orbital or magnetic field models. Indeed, orbital information are extracted by processing a magnetic signal with length of several orbits. It results in semi-major axis estimates that represent average values over the signal timespan and gives the method robustness against sensor noise or calibration errors. Moreover, the procedure needs only the knowledge of the magnetic field strength. It is therefore independent of the satellite's attitude. Finally, the procedure does not require an absolute time information, which allows one to apply the method even when the onboard clock is not synchronized. However, the procedure evidences a moderate sensitivity to the onboard clock drift, which may require special care while designing the time measurement unit.

The presented procedure, validated through numerical laboratory and flight data, represents a significant enhancement of the original work presented in \cite{11,12}. It now yields semi-major axis estimates with errors generally far below 1 km. This work intends
to be a first step toward the development of an extended algorithm that attempts at estimating all six orbital elements from the spectral analysis of the magnetic signal. Nonetheless, it is believed that the results obtained so far already have some potential applications. For instance, the procedure could be used for the initialization of Kalman filter-based methods. It could also be exploited when many small satellites of a cluster are released at the same time, which poses the problem of object identification by the operator [13].

The paper is structured as follows. Section 2 provides the theoretical background behind the proposed method. Section 3 illustrates the algorithm used for the identification of the semi-major axis. In Section 4 several application cases validating the procedure are presented. They involve numerical simulations, laboratory experiments and actual flight data analyses. Conclusions and future work are given in Section 5.

2. Theoretical background

This section intends to give a rationale of the method from a mathematical perspective. We firstly recall mathematical properties of the model used to describe the Earth’s magnetic field in LEO. Then, the periodic properties of the satellite’s motion are examined for different orbital models. Investigating the effect of spectral features of the orbital motion on the magnetic field model allows us to explain the spectral behavior of the onboard magnetic field measurement. A Keplerian orbit model is initially examined because this simplified framework offers a better understanding of basic principles of the method. Equipped with this knowledge the analysis moves toward a model including most important perturbation effects. The case of more sophisticated models and real scenario are addressed at the end of the section.

The position of a satellite can be described through the position vector \( \mathbf{r} \) applied to the Earth’s center. This vector can be represented in the Earth-Centered Inertial (ECI) frame with axes \( X, Y \) and \( Z \) or in the Earth-Centered Rotating (ECR) frame with axes \( x, y \) and \( z \). The two frames are rotated to each other along the common \( z = Z \) axis by an angle \( \delta \) variable with the time. In the ECR frame the position vector can be expressed in spherical coordinates that are the radius \( r \), the longitude \( \phi \) and the co-latitude \( \theta \). Alternatively, the satellite’s motion can be examined through a set of Keplerian orbital elements. These elements define the shape of the orbit, the position of the orbit with respect to ECI and the position of the satellite in the orbit. The six orbital elements we use in this work are the semi-major axis, \( a \), the inclination, \( i \), the eccentricity, \( e \), the right ascension of ascending node, \( \Omega \), the argument of perihelion, \( \omega \) and the true anomaly, \( v \). These elements may be clearly visualized by introducing a perifocal frame with axes \( \bar{x}, \bar{y} \) and \( \bar{z} \). All parameters introduced in this paragraph are depicted in Fig. 1.

2.1. Earth’s magnetic field model

An accurate description of the Earth’s magnetic field in LEO is given by the spherical harmonic model proposed by the International Association of Geomagnetism and Aeronomy (IAGA) [14] called the International Geomagnetic Reference Field (IGRF) model. It takes the following expression

\[
V(r, \phi, \theta) = \sum_{k=1}^{\infty} \left( \frac{R}{r} \right)^{u+1} \sum_{v=0}^{u} \left( g^v_u \cos v \phi + h^v_u \sin v \phi \right) \frac{\partial P^v_u}{\partial \theta} (\cos \theta), \tag{1}
\]

where \( V \) is the magnetic potential, \( k \) the order of the approximation and \( R \) the reference Earth’s radius. \( P^v_u (\cos \theta) \) terms are the Schmidt quasi-normalized associated Legendre functions. The model is parametrized by the \( g^v_u \) and \( h^v_u \) coefficients that slightly change over time and undergo major update every 5 years by IAGA. In our analysis we consider \( g^v_u \) and \( h^v_u \) constant with the time. The magnetic field vector \( B \) can be derived from the scalar potential as follows [15]:

\[
B_r = -\frac{\partial V}{\partial r} = \sum_{k=1}^{\infty} \left( \frac{R}{r} \right)^{u+1} \sum_{v=0}^{u} \left( g^v_u \cos v \phi + h^v_u \sin v \phi \right) \frac{\partial P^v_u}{\partial \theta} (\cos \theta), \tag{2}
\]

\[
B_\theta = -\frac{1}{r} \frac{\partial V}{\partial \theta} = \sum_{k=1}^{\infty} \left( \frac{R}{r} \right)^{u+1} \sum_{v=0}^{u} \left( g^v_u \cos v \phi + h^v_u \sin v \phi \right) \frac{\partial P^v_u}{\partial \theta} (\cos \theta), \tag{3}
\]

\[
B_\phi = -\frac{1}{r \sin \theta} \frac{\partial V}{\partial \phi} = \sum_{k=1}^{\infty} \left( \frac{R}{r} \right)^{u+1} \sum_{v=0}^{u} \left( g^v_u \cos v \phi + h^v_u \sin v \phi \right) \frac{\partial P^v_u}{\partial \theta} (\cos \theta), \tag{4}
\]

where \( B_r, B_\theta \) and \( B_\phi \) are the \( B \) components in the North–East Down reference frame at the satellite’s location. Expanding, for in-
stance, Equation (2) and assuming for the sake of simplicity that $k = 1$ we have

$$B_r = \frac{A \cos \theta + B \sin \theta (C \cos \phi + D \sin \phi)}{r^3}$$

(5)

with $A$, $B$, $C$ and $D$ coefficients dependent only on constant terms. Their expressions are not given here as not relevant in this treatment. For the purpose of this study it is important to highlight only the particular analytic form taken by $B_r$. It involves products and divisions of $\theta$ and $\phi$ trigonometric operators and $r$. Very similar expressions can be obtained for $B_\theta$, $B_\phi$ and $B_\psi$, with this latter being the square of the magnetic field magnitude. Equations from (2) to (4) can be used to model the magnetic field experienced by an orbiting satellite. In this case $r$ and sines and cosines of $\theta$ and $\phi$ are time-dependent functions that possess well-defined periodic properties. In the next subsections we will show how these spectral features affect the spectral behavior of $B^2$ for orbital models of increasing complexity.

2.2. Keplerian model

Firstly, we introduce the frequencies $f_E$ and $f_{sat}$. The former is the frequency of rotation of ECR with respect to ECI. The latter is the frequency of revolution of the orbiting body. It is related to the mean motion, $n$, through

$$f_{sat} = \frac{1}{T_{sat}} = \frac{n}{2\pi},$$

(6)

where $T_{sat}$ the orbital period. In a Keplerian model $f_{sat}$ is also the fundamental frequency of $r$ (except for the case of circular orbits where $r$ is obviously constant), which is a periodic function with spectral components at $f_{sat}$ harmonics. The Fast Fourier Transform (FFT) amplitude spectrum of $r$ for a non-circular orbit is depicted in Fig. 2a. The satellite co-latitude is also a periodic function with fundamental frequency $f_{sat}$. The analytical dependency of $\sin \theta$ and $\cos \theta$ on $f_{sat}$ for the particular case of circular orbits is proven in Appendix A by Equations (28) and (29). For elliptic orbits the Fourier amplitude diagram is enriched of $f_{sat}$ harmonics, as visible in Fig. 2b. The analysis of the longitude needs to account for the fact that when the satellite has completed one orbit in the inertial frame, the Earth has also rotated by a certain angle $\delta$ meanwhile. Thus, the frequency contents of $\sin \phi$ and $\cos \phi$ are related to both $f_{sat}$ and $f_E$. Equations (30) and (31) in Appendix A point out signal modulations existing between $f_{sat}$ and $f_E$ in the expressions of $\sin \phi$ and $\cos \phi$ when the orbit is circular and its inclination different from zero. In the case of elliptic orbits (with inclination different from zero) these modulations take place also among $f_E$ and $f_{sat}$ harmonics. It produces a Fourier spectrum with peaks at linear combinations of $f_{sat}$ and $f_E$, as depicted in Fig. 2c.

Considering the frequency contents of $r$ and sines and cosines of $\theta$ and $\phi$ and recalling that $B^2$ is composed of sums of products and divisions of these functions, it turns out that the frequency behavior of $B^2$ is the result of a large number of modulations (increasing with the order of the IGRF model) between harmonics of $f_{sat}$ and frequencies that are linear combinations of $f_{sat}$ and $f_E$. Thus, the Fourier diagram of $B^2$ will show numerous frequency peaks located at well-defined frequencies with values $N_s f_{sat} \pm N_E f_E$, where $N_s$ and $N_E$ are natural numbers. The peaks have different amplitudes which increase when moving toward frequencies that are multiple of $f_{sat}$, as depicted in Fig. 3. The shape of the spectrum suggests that $f_{sat}$ can be determined easily by a peak picking procedure. Eventually, the orbit’s semi-major axis can be evaluated from $f_{sat}$ by the third Kepler’s law

$$f_{sat} = \frac{\sqrt{\mu}}{2\pi a^2},$$

(7)

with $\mu$ denoting the Earth’s gravitational constant.

2.3. Secular J2 model

The secular J2 model represents a significant improvement when passing from the ideal Keplerian model to a more realistic model with perturbations. The most important perturbation in LEO is due to the Earth’s oblateness, which induces gravitational effects that deviate the satellite’s orbit. Similarly to the IGRF model, also the Earth’s gravitational perturbation can be described by a spherical harmonic model. The model accuracy is controlled by zonal harmonics of increasing order. In the secular J2 model only the secular effect of the first non-zero zonal harmonic with coefficient $J_2$ is considered, which is by far the most important term. The $J_2$ harmonic perturbs all orbital elements. Using variational Gauss equations it is possible to obtain analytic expressions for the rates of changes of the osculating orbital elements [16]. When filtering short-period variations by averaging the rates of variations of the orbital parameters over an orbit period, only the secular variations are preserved. Thus, the secular rates can be written as

$$[\dot{a}]_{J2} = [\dot{\hat{a}}]_{J2} = 0$$

(8)

$$[\dot{e}]_{J2} = [\dot{\hat{e}}]_{J2} = 0$$

(9)

$$[\dot{i}]_{J2} = [\dot{\hat{i}}]_{J2} = 0$$

(10)
The J2 Equations

The J2 model is used to model the perturbations due to the oblateness of the Earth. The J2 term is given by

\[ \mathbf{J}_2 = \frac{3}{2} \frac{R^2 J_2}{\bar{a}^2(1 - e^2)^2} \cos i \]

where \( \bar{a} \) is the semi-major axis of the satellite's orbit, \( R \) is the radius of the Earth, \( J_2 \) is the second zonal harmonic, \( e \) is the eccentricity, and \( i \) is the inclination of the satellite's orbit.

The secular variations of the elements due to the J2 perturbations are given by

\[ \dot{\Omega}_{sJ2} = -\frac{3}{2} \frac{R^2 J_2}{\bar{a}^2(1 - e^2)^2} \cos i \]

\[ \dot{\omega}_{sJ2} = -\frac{3}{4} \frac{R^2 J_2}{\bar{a}^2(1 - e^2)^2} (5 \cos^2 i - 1) \]

\[ \dot{\bar{m}}_{sJ2} = \frac{3}{4} \frac{R^2 J_2}{\bar{a}^2(1 - e^2)^2} (3 \cos^2 i - 1) \]

Equations from (8) to (13) use the notation [·]_{sJ2} for referring to perturbed elements of the J2 model.

The secular variations of the fundamental frequencies are defined as

\[ f_m = \frac{\dot{\Omega}_{sJ2}}{2\pi} = \frac{3}{2} \frac{R^2 J_2}{\bar{a}^2(1 - e^2)^2} \cos i \bar{f}_{sat} = C_m \bar{f}_{sat} \]

\[ f_a = \frac{\dot{\omega}_{sJ2}}{2\pi} = -\frac{3}{4} \frac{R^2 J_2}{\bar{a}^2(1 - e^2)^2} (5 \cos^2 i - 1) \bar{f}_{sat} = C_a \bar{f}_{sat} \]

\[ f_\Delta = \frac{\dot{\bar{m}}_{sJ2}}{2\pi} = \frac{3}{4} \frac{R^2 J_2}{\bar{a}^2(1 - e^2)^2} (3 \cos^2 i - 1) \bar{f}_{sat} = C_\Delta \bar{f}_{sat} \]

where \( f_m, f_a \), and \( f_\Delta \) are the fundamental, apsidal, and mean anomaly variations, respectively. These variations can be used to predict the motion of a satellite in orbit.

The spectral behavior of the elements can be examined by comparing their FFT diagrams. The FFT of the fundamental frequency of the satellite's motion is shown in Fig. 4a. The FFT of the apsidal frequency is shown in Fig. 4b. The FFT of the mean anomaly frequency is shown in Fig. 4c.

From these FFT diagrams, it is possible to observe the presence of harmonics and the modulation of the frequencies over time. The spectral analysis of the elements can provide valuable information about the dynamics of the satellite's orbit and the perturbations it experiences.
Equation (15) that its absolute value does not exceed $6 \cdot 10^{-7}$ Hz in orbits with altitudes above 200 km, then all peaks are featured provided that the resolution of the spectrum is very high, that is the acquisition time is very long. An example of $B^2$ Fourier diagram for two signals of very different time lengths is presented in Fig. 5. In the case of long acquisition time it is possible to distinguish all spectral components of the signal. Instead, near peaks are merged in a wider lobe in the case of short signal. As we will see in the next section the identification procedure is based on accurate estimates of $f_a$, $f_0$ and $f_m$ from the FFT amplitude diagram of $B^2$. This operation is challenging in real applications where, for practical reasons, the length of the signal has to be limited.

2.4. Real scenario

Without secular simplifications, perturbations cause all orbital elements to change over time. This means that the characteristic frequencies introduced in the secular $J2$ model are not constant anymore. In principle in these conditions Fourier analysis has strong limitations because it is not able to correlate frequency variations with time (here the Wavelet or Hilbert transforms could probably be useful). However, it has been observed that the Fourier diagram of $B^2$ remains qualitatively the same in case of real data because the characteristic frequencies undergo only very small changes during the time of acquisition. It is believed that modifications of the Fourier spectrum should only concern with a small widening and shift of the lobes, in agreement with the known behavior of the Fourier analysis of non-stationary signals.

3. Identification procedure

The identification procedure described in this section has been developed assuming a secular $J2$ model and LEOs with eccentricities lower than 0.1 and inclinations below 120 deg. Numerical analyses have also shown loss of accuracy for values of inclination and eccentricity above the aforementioned limits. A discussion on the applicability of the procedure to more complicated orbital models or to flight data is addressed at the end of this section and validated experimentally in Section 4.3.

3.1. Fourier analysis of the magnetic field magnitude

The identification procedure examines the amplitude of the FFT of $B^2$ herein denoted by $|\mathcal{F}(B^2)|$. The length of the elaborated magnetic signal is $T_{\text{facq}}$ and the sampling frequency is $f_{\text{facq}}$. These two parameters control, to a certain extent, the accuracy of the estimate. The choice of $T_{\text{facq}}$ and $f_{\text{facq}}$ is empirically made after having assessed their influence on the error of the estimates. It turns out that for $T_{\text{facq}}$ above 70 orbits and $f_{\text{facq}}$ above $2 \cdot 10^{-3}$ Hz the procedure performance gets stable. In the following analyses, unless explicitly said, we use values $T_{\text{facq}} = 80T_{\text{sat}}$ and $f_{\text{facq}} = 0.01$ Hz. The limited extent of $T_{\text{facq}}$ produces significant leakage in the FFT. In order to reduce its effect a Blackman window is applied to the signal in the time-domain. A 0-padding technique is also employed to artificially increase the frequency resolution of the spectrum.

3.2. Evaluation of the semi-major axis from the frequency spectrum

As explained in Section 2.3, $|\mathcal{F}(B^2)|$ diagrams of secular $J2$ orbits exhibit lobes at frequencies $N_m f_m \pm N_a f_0 \pm N_A f_A$. However, when $T_{\text{facq}} = 80T_{\text{sat}}$ many near lobes, visible with longer acquisition times, are merged in a unique lobe whose peak is located at a frequency which, in general, is not equal to any of the frequencies $N_m f_m \pm N_a f_0 \pm N_A f_A$. An algorithm that effectively operates in these conditions has been developed. The algorithm needs a rough guess of the semi-major axis, $\hat{a}$, and imprecise knowledge of the inclination, $i$, and eccentricity, $e$. With reference to the diagrams in Figs. 6 and 7, the algorithm’s steps are described below:

Algorithm.

a) Calculate the initial guess of $f_{\text{sat}}$, $\tilde{f}_{\text{sat}}$, from $\hat{a}$ using Equation (7). The value of $\hat{a}$ may be largely imprecise, provided that the following condition is met.

$$|\tilde{f}_{\text{sat}} - f_{\text{sat}} - f_n| < \frac{f_A}{2}.$$  \hspace{1cm} (17)

Equation (17) cannot be checked with exactness in this phase because the involved frequencies are still unknown. However, it is possible to show that it is satisfied for $\hat{a}$ errors of several dozens of kilometers. In this work it is assumed that $\hat{a} = a + 50$ km.

b) Search for the nearest lobe to $\tilde{f}_{\text{sat}} - f_0$. This reference lobe is labeled $L_{\text{ref}}$. The utilization of a reference lobe that is not the one associated with $f_{\text{sat}}$ is motivated by the fact that this latter becomes very small when the inclination approaches zero. Since small lobes may cause significant errors in the calculation of the central frequency addressed in the next point, the current choice of the reference lobe guarantees that the procedure holds the same level of accuracy even for orbits with small inclinations.

c) Calculate the central frequency of $L_{\text{ref}}$, $f_{\text{Lref}}$, using
where \( f_i \) and \( F_i \) are the frequencies and FFT amplitudes of the set of point \( I \) forming \( L_{\text{ref}} \). The \( I \) points must also fulfill the condition to stay above a defined threshold. The threshold is used for filtering out possible signal noise appearing at low \( F(B^2) \) amplitudes. Increasing the frequency resolution by 0-padding ensures that \( F_I \) calculated by Equation (18) is located at the mid of the lobe.

d) Provide a first estimate of the satellite frequency by \( \hat{f}_{\text{sat}} = f_{\text{sat}} + f_E \) and calculate the semi-major axis, \( \hat{a} \), using Equation (7).

e) From \( \hat{a}, \hat{i} \) and \( \hat{e} \) calculate \( C_a, C_m \) and \( C_n \) given in Equations (14), (15) and (16).

f) Improve the \( f_{\text{sat}} \) estimate. It has been observed that the following property holds

\[
f_{\text{sat}} \simeq f_m + x_a f_n - f_\Delta,
\]

where \( x_a \) is an empirical coefficient which depends on the orbit inclination and eccentricity. Thus, plugging Equations (14), (15) and (16) into Equation (19) gives an improved estimate of \( f_{\text{sat}} \) by

\[
\hat{f}_{\text{sat}} = \frac{f_{\text{sat}} + f_E}{C_m - C_n + x_a C'_a},
\]

where \( C'_a \) denotes \( C_a \) without the part dependent on the inclination and eccentricity, namely

\[
C'_a = \frac{3}{2} \frac{R^2 J_2}{a^2}.
\]

g) Calculate the final estimate of the semi-major axis, \( \hat{a} \), from \( \hat{f}_{\text{sat}} \).

The utilization of the coefficient \( x_a \) accounts for the fact that \( f_{\text{sat}} \) is not located exactly at \( f_m - f_\Delta \) but it is shifted of a portion of \( f_n \) depending on the orbit’s inclination and, to a small extent, on the orbit’s eccentricity. The relationship existing between \( x_a \) and \( i \) and \( e \) is smooth and can be accurately described by a fifth-order polynomial function, \( P_5(i, e) \). Fig. 8 shows interpolated \( x_a \) values calculated on a regular \( i-e \) grid by optimization processes that minimize the absolute semi-major estimate error \( |\hat{a} - a| \). Numerical analyses have shown that the correction due to the utilization of Equation (20) results in significant improvements of the estimates with respect to \( \hat{f}_{\text{sat}} \). An explanation of the particular \( x_a \) behavior and its negligible dependency on the other orbital elements is a subject of current investigation. Conveniently, \( \hat{a} \) shows also great robustness against incorrect \( \hat{i} \), \( \hat{a} \), and \( \hat{e} \). It is analytically proven in Appendix B and confirmed in Section 4.1 via numerical analyses.

3.3. More advanced models and real scenarios

The identification algorithm presented above is effective also when applied to more complicated perturbation models or real flight data where higher zonal harmonics and atmospheric drag that induce variations of all six orbital elements. Only the atmospheric drag sensibly affects the semi-major axis producing a orbit decay that, in general, is not larger than a very few meters per orbit. In these cases, the application of the algorithm yields estimates that tend to the average of the mean semi-major axis over \( T_{\text{acq}} \).

4. Application cases

The performance of the procedure presented in Section 3 has been tested by numerical simulations, laboratory experiments and data collected from ongoing satellite missions. Obtained results are illustrated in the following subsections.

4.1. Numerical model

Numerical simulations provide a handy framework for testing the performance of the procedure on a large set of orbits. Simulations of the satellite’s orbit and magnetic field have been carried out within Scilab software. Scilab embeds the CelestLab library.
(version 3.1.0) developed by the French Space Agency. The library incorporates the IGRF model and four different closed-form analytic orbit models: Keplerian, Secular J2 [16], Lyddane [18,19] and Eckstein–Hechler [20]. Last three models implement the Earth’s oblateness perturbation but do not include drag effects. Tests with all four models have been conducted. Since Lyddane and Eckstein–Hechler models yielded almost identical outcomes, then only results from the secular J2 and Lyddane models are presented in the paper. The secular J2 model is initialized by providing osculating orbital elements \([a_0, \epsilon_0, i_0, \Omega_0, \omega_0, M_0]\). As already said, this model does not perturb the semi-major axis and therefore \(a\) remains constant in the simulation. The Lyddane model is initialized by mean orbital elements \([\bar{a}_0, \bar{\epsilon}_0, \bar{i}_0, \bar{\Omega}_0, \bar{\omega}_0, \bar{M}_0]\) and calculates both osculating and mean elements at specified time instants. The propagated mean semi-major axis is constant due to the lack of drag in the model, while its osculating counterpart is variable with the time and oscillates around the mean element.

Two different kinds of analyses have been performed: Monte Carlo analysis and analysis of the influence of \(i\) and \(\bar{i}\) errors on the estimates accuracy. The Monte Carlo simulation involved 300 random orbits with initial orbital elements defined within the ranges given in Table 1. The maximum orbit inclination was limited to 120 deg. Larger values of inclinations are very rare in small satellite missions. With the setup given in Section 3.1 the number of samples of the magnetic signals varied between 4300 and 6200. The signal length is then extended to 131072 after applying 0-padding. The accuracy of the estimates of the identification procedure is presented by histograms of \(\hat{a} - a_0\) and \(\hat{\bar{a}} - \bar{a}_0\) in Figs. 9a and 9b for the secular J2 and Lyddane models, respectively. The histograms show similar results for the two models with absolute errors far below 1 km and standard deviations below 0.1 km.

Results presented in Fig. 9 have been obtained assuming correct values for \(i\) and \(\bar{i}\). In practice, these values may be unknown. Thus, the robustness of the procedure to wrong values of \(i\) and \(\bar{i}\) has been verified. The analysis is conducted on a Lyddane orbit by altering the two parameters \(i\) and \(\bar{i}\) one at the time. In particular, the inclination is perturbed of 1 deg and 2 deg from its right value \(i_0\), while a variation of 20% of the right value \(\epsilon_0\) is applied to the eccentricity. Results are presented in Table 2 in terms of \(\bar{a}'\) and \(\hat{\bar{a}}'\) for four orbits with different inclination. They show a negligible influence of \(\bar{\epsilon}\) mismatch for all orbits. On the other hand, the estimate error due to \(i\) inaccuracy becomes appreciable for orbits with inclination of about 50 deg. These results are in agreement with the sensitivity analysis presented in Appendix B.

### 4.2. Hardware-in-the-loop experiments

Hardware-in-the-loop experiments have been conducted in the AGH Space Tech Lab. The lab is equipped with a Helmhotz cage producing a controllable magnetic field and a spherical air bearing that enables free rotations of the test article mounted on its top. In this application the test article is composed of a Freescale KL46Z board embedding a low-cost MAG3110 3-axis magnetometer. A picture of the test rig is shown in Fig. 10. The magnetic field signal created with numerical models is sent to the programmable power supplies that energize the three pairs of coils of the cage with an update rate of 1 Hz. The magnetic field is measured by the sensor of the KL46Z board and stored on a pendrive for post-processing. Using a time scaling of 100:1 (100 s of real flight correspond to 1 s in the experiment) and with the setup described in Section 3.1 the duration of an experiment is reduced to about two hours. The MAG3110 measurement is a noisy replicate of the original IGRF signal sent to the coils controller. The scatter is mainly due to sensor noise (±0.5 μT) and calibration issues affecting the cage and the test article (about ±1 μT). A visual comparison of a short portion of the original numerical IGRF signal \((B_{num})\) with the MAG3110 measurement \((B_{lab})\) is given in Fig. 11 (the plot includes also a third signal whose meaning is described in Section 4.3). The agreement between the two signals can be assessed by the Root Mean Square (RMS) error measured over the full length of the signal. Seven experiments with secular J2 and Lyddane models have been carried out. In the beginning three replicates of the same secular J2 orbit have been conducted for checking the measurement repeatability. Then, the same orbital elements have been propagated using the Lydanne model. Experiments with the orbits simulating the two missions described in Section 4.3 were also performed. The experimental setup and outcomes of each test are presented in Table 3. Here, the header \(\epsilon_{RMS}\) denotes the RMS error between \(B_{num}\) and \(B_{lab}\), while \(\Delta_{num}\) and \(\Delta_{lab}\) denote the semi-major axis estimates from \(B_{num}\) and \(B_{lab}\), respectively. The level of accuracy of \(\Delta_{lab}\) and \(\Delta_{num}\) is the same and it does not differ from that obtained in the previous section. Moreover, the errors are not correlated to \(\epsilon_{RMS}\). These outcomes prove

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**Table 1** Ranges of variability of the orbital elements used in the Monte Carlo simulation.

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<tr>
<th>Parameter</th>
<th>Unit</th>
<th>Lower bound</th>
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<tbody>
<tr>
<td>(a_0)  ((\hat{a}_0))</td>
<td>[km]</td>
<td>6600</td>
<td>8440</td>
</tr>
<tr>
<td>(\epsilon_0) ((\hat{\epsilon}_0))</td>
<td>[-]</td>
<td>0</td>
<td>0.1</td>
</tr>
<tr>
<td>(i_0) ((\hat{i}_0))</td>
<td>[deg]</td>
<td>0</td>
<td>120</td>
</tr>
<tr>
<td>(\Omega_0) ((\hat{\Omega}_0))</td>
<td>[deg]</td>
<td>0</td>
<td>90</td>
</tr>
<tr>
<td>(\omega_0) ((\hat{\omega}_0))</td>
<td>[deg]</td>
<td>0</td>
<td>90</td>
</tr>
<tr>
<td>(M_0) ((\hat{M}_0))</td>
<td>[deg]</td>
<td>0</td>
<td>50</td>
</tr>
</tbody>
</table>

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![Fig. 8](image-url) Plots showing the dependency of \(x_a\) on the orbit’s inclination and eccentricity within the ranges 0–130 deg and 0–0.1, respectively. The function \(x_a(i, \epsilon)\) can be accurately approximated by a 5th order polynomial function.

![Fig. 9](image-url) Plots showing the dependency of \(x_a\) on the orbit’s inclination and eccentricity within the ranges 0–130 deg and 0–0.1, respectively. The function \(x_a(i, \epsilon)\) can be accurately approximated by a 5th order polynomial function.
Fig. 9. Histograms of the semi-major axis prediction errors when using (a) the secular J2 orbit propagator and (b) the Lyddane orbit propagator.

Table 2
Semi-major axis estimates resulting from deviating $\hat{e}$ of 20% and $\hat{i}$ of 1 deg and 2 deg from their correct values. All semi-major axis values are expressed in km and inclinations in deg.

<table>
<thead>
<tr>
<th>Test Nr.</th>
<th>$\hat{a}_0$</th>
<th>$\hat{e}_0$</th>
<th>$\hat{i}_0$</th>
<th>$\hat{a}_{1b1}$</th>
<th>$\hat{a}_{2b1}$</th>
<th>$\hat{a}_{3b1}$</th>
<th>$\hat{a}_{4b1}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>7450.00</td>
<td>0.1</td>
<td>45</td>
<td>Secular J2</td>
<td>1</td>
<td>0.53</td>
<td>7450.07</td>
</tr>
<tr>
<td>2</td>
<td>7450.00</td>
<td>0.1</td>
<td>45</td>
<td>Secular J2</td>
<td>2</td>
<td>0.50</td>
<td>7450.07</td>
</tr>
<tr>
<td>3</td>
<td>7450.00</td>
<td>0.1</td>
<td>45</td>
<td>Secular J2</td>
<td>3</td>
<td>0.93</td>
<td>7450.07</td>
</tr>
<tr>
<td>4</td>
<td>7450.00</td>
<td>0.1</td>
<td>45</td>
<td>Lyddane</td>
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<td>0.58</td>
<td>7449.93</td>
</tr>
<tr>
<td>5</td>
<td>6840.20</td>
<td>6.2e-4</td>
<td>87.35</td>
<td>Lyddane</td>
<td>1</td>
<td>0.74</td>
<td>6840.32</td>
</tr>
<tr>
<td>6</td>
<td>7013.20</td>
<td>7.3e-3</td>
<td>97.71</td>
<td>Lyddane</td>
<td>1</td>
<td>0.87</td>
<td>7013.21</td>
</tr>
</tbody>
</table>

Fig. 10. Laboratory test rig for simulating satellite onboard measurements of the magnetic field. The test article is equipped with a low-cost magnetometer and floats upon a spherical air bearing located within a Helmholtz cage.

Table 3
Semi-major axis estimates calculated from $B_{num}$ and $B_{lab}$. All semi-major axis values are expressed in km, inclinations in deg and RMS errors in $\mu$T.

<table>
<thead>
<tr>
<th>Test Nr.</th>
<th>$\hat{a}_{num}$</th>
<th>$\hat{a}_{lab}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>7450.07</td>
<td>7450.09</td>
</tr>
<tr>
<td>2</td>
<td>7450.07</td>
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<td>6840.32</td>
<td>6840.27</td>
</tr>
<tr>
<td>6</td>
<td>7013.21</td>
<td>7013.25</td>
</tr>
</tbody>
</table>
the robustness of the procedure to different kinds of signal error present in a physical framework.

4.3. Satellite flight data

Finally, the identification procedure has been validated by flight data collected from SWARM and UWE-3 missions. The SWARM satellite constellation, designed by the European Space Agency, aims to study the Earth magnetosphere. Three identical satellites are equipped with high-resolution sensors measuring strength and direction of the magnetic field, as well as precise navigational package. Two of the satellites, SWARM-A and SWARM-C, trail each other on an orbit with altitude of about 460 km and an inclination of 87°. The third satellite, SWARM-B, occupies a 530 km orbit with almost the same inclination. SWARM offers very precise measurements of the magnetic field. The UWE-3 mission has a completely different character. It involves a 1U CubeSat nanosatellite designed and operated by the University of Würzburg [21]. Its altitude, inclination and eccentricity are 636 km, 98 deg and 0.007, respectively. The satellite is equipped with a low power real-time attitude determination and control system [22]. The system features 9 single-axis magnetometers, of which the primary three sensors were used for this analysis. Measurement inaccuracies include sensor noise and uncertainties in the sensor calibration such as axes cross-talk due to not precise 3-axis perpendicular mounting, gain and offset irregularities caused by the physically different sensor units and the presence of hard and soft magnetic materials inside the satellite. The sensors have been calibrated in orbit proving a level of noise of about 1.5 μT (3σ variance). Differently from all other cases, UWE-3 data use $T_{eq} = 72T_{sat}$ and $f_{eq} = 1/58$ Hz.

Onboard magnetic measurements with magnitude $B_{sat}$ have also been simulated numerically using the Lyddane model initialized with the real mean orbital elements obtained from the satellite’s tracking system. For two cases, SWARM-A and UWE-3, the numerical signal has also been processed in the lab (tests Nr. 5 and 6 in Table 3), as explained in the previous section. Numerical and laboratory magnetic signals evidence a good matching with flight data, as shown in Fig. 11 where a short fragment of the three signals is depicted.

The semi-major axis estimates the identification procedure have been confronted with the real values obtained from TLE and satellite position data. For SWARM constellation, accurate onboard measurements of the satellites’ position were used for determining the mean semi-major axis by means of CelestLab procedures. Instead, orbital elements of UWE-3 have been calculated from TLE obtained from the CelestTrack service. This latter approach is expected to be less precise than the former one. The evolutions of the real mean semi-major axis, $\hat{a}_{sat}$, are presented in the plots of Fig. 12. They highlight the orbit decay caused by atmospheric drag, which affects both missions, but to a different extent. The application of the identification procedure to orbits with decreasing altitude results in underestimates of the initial semi-major axis $\hat{a}_{sat}$. This behavior is visible in the diagrams of Figs. 12 from (a) to (c) for which the orbit decay is more significant. Here, since $\hat{a}_{sat}$ is always below $\hat{a}_{num}$ – with $\hat{a}_{num}$ obtained from a numerical model that does not account for drag effects – then it is believed that the negative gap is caused by the increasing frequencies of the $B_{sat}$ signal caused by the orbit decay. Different conclusions are drawn from the examination of the UWE-3 plot. Here the drag effect is very small and the difference between $\hat{a}_{sat}$ and $\hat{a}_{stat}$ should be only a consequence of the lack of accuracy of the procedure. The estimated error is about 700 m, which is sensibly larger than any other value obtained from previous analyses. It is believed that two different factors might cause this lost of performance: inaccuracies in the assessment of $\hat{a}_{sat}$ from TLE or in the TLE data itself, imprecision of the UWE-3 onboard clock that may affect the $T_{eq}$ determination, which leads in turn to a scaling of the FFT spectrum (it has been calculated that for any second of $T_{eq}$ offset $\hat{a}$ varies of about 10 m). All semi-major axis estimates presented in this section are summarized in Table 4.

5. Conclusions

A novel method for estimating the semi-major axis of a satellite’s orbit has been presented. The method is based on the frequency analysis of the onboard magnetic field measurements. Its main advantages are the autonomy, the need for magnetic measurements solely, the independence of the satellite attitude and the fact it does not require magnetic or orbital models. The procedure has been tested numerically by Monte Carlo simulations on a wide range of realistic LEOs using several kinds of orbital propagators. Numerical results show that orbit semi-major axis can be estimated with an average error of about 100 m and that rarely exceeds 500 m. Resilience of the algorithm against magnetic field measurement errors due to sensor noise, calibration issues, sample corruption or imprecise sampling, has also been proven by
hardware in the loop experiments and using real flight data. Furthermore, since the procedure is simple it is thought that its future onboard implementation is not very cumbersome and the required computational burden essentially consists of a FFT applied to a signal of about 130000 samples.

There are several topics open for future investigations. First of all, since the procedure yields an average of the semi-major axis over the acquisition time rather than the mean or osculating element at the current satellite's position of the satellite, then a link with these latter should be found for satellite's position determination purposes. Further, the performance of the procedure is well understood for secular J2 orbits but more work is needed to better comprehend some theoretical aspects behind the method and to extensively validate the procedure in case of models that include drag effects or real data. In this regard, also different kinds of spectral analyses are thought to be useful, like the Wavelet or the Hilbert transforms. Ongoing work also deals with the identification of other orbital parameters by exploiting more deeply information included in the amplitude and phase Fourier diagrams.

**Conflict of interest statement**

The authors declare that there is no conflict of interest.

**Appendix A**

In order to understand the spectral behavior of sines and cosines of $\theta$ and $\phi$ it is convenient to make their dependency on $f_E$ and $f_{sat}$ explicit for the particular case of Keplerian circular orbits. For this purpose we need to write the rectangular components of the satellite's position vector $\mathbf{r} = \mathbf{r}_E$ calculated in ECR as functions of the angles $\delta, \omega, \nu$ and $i$ already defined in Section 2, as follows:

$$r_x = r [\cos \delta \cos(\omega + \nu) + \sin \delta \sin(\omega + \nu) \cos i]$$  \hspace{1cm} (22)
$$r_y = r [-\sin \delta \cos(\omega + \nu) + \cos \delta \sin(\omega + \nu) \cos i]$$  \hspace{1cm} (23)
$$r_z = r \sin(\omega + \nu) \sin i.$$  \hspace{1cm} (24)

Equations (22)–(24) are obtained by applying a ECI-to-ECR frame transformation [16]. The projection of $\mathbf{r}$ on the $xy$ plane of ECR further defines the $f_{xy}$ component

$$r_{xy} = r \sqrt{\cos^2(\omega + \nu) + \sin^2(\omega + \nu) \cos^2 i}.$$  \hspace{1cm} (25)

Next, we express the angles $\nu$ and $\delta$ in terms of $f_E$ and $f_{sat}$ as follows:

$$\nu = nt = 2\pi f_{sat} t$$  \hspace{1cm} (26)

$$\delta = \phi - \frac{2\pi}{f_E} t$$  \hspace{1cm} (27)
\[ \delta = \omega_E t = 2\pi f_{E} t, \]  

(27)

where \( \omega_E \) is the angular velocity of the Earth. In Equations (26) and (27) we assumed that both \( \nu \) and \( \delta \) are zero when \( t \) is zero. If we plug Equations (26) and (27) into Equations (22)–(25), set \( \omega = 0 \) and express sines and cosines of \( \theta \) and \( \phi \) as ratios of \( r \) components, then we have:

\[
\cos \theta = \frac{r_z}{r} = \sin(2\pi f_{sat} t) \sin i
\]

(28)

\[
\sin \theta = \frac{r_{xy}}{r} = \sqrt{\cos^2(2\pi f_{sat} t) + \sin^2(2\pi f_{sat} t)} \cos^2 i
\]

(29)

\[
\cos \phi = \frac{r_x}{r_{xy}} = \frac{\cos(2\pi f_{E} t) \cos(2\pi f_{sat} t) + \sin(2\pi f_{E} t) \sin(2\pi f_{sat} t) \cos i}{\sqrt{\cos^2(2\pi f_{sat} t) + \sin^2(2\pi f_{sat} t) \cos^2 i}}
\]

(30)

\[
\sin \phi = \frac{r_y}{r_{xy}} = \frac{-\sin(2\pi f_{E} t) \cos(2\pi f_{sat} t) + \cos(2\pi f_{E} t) \sin(2\pi f_{sat} t) \cos i}{\sqrt{\cos^2(2\pi f_{sat} t) + \sin^2(2\pi f_{sat} t) \cos^2 i}}
\]

(31)

Equations (28) and (29) say that \( \sin \theta \) and \( \cos \theta \) have identical spectral properties with a single component at \( f_{sat} \). The spectral components of \( \sin \phi \) and \( \cos \phi \) are also equal to each-other. In both cases they result from the same modulation existing between \( f_E \) and \( f_{sat} \), as shown by Equations (30) and (31).

For elliptical orbits the true anomaly does not change linearly with the time therefore Equation (26) is not valid. Nonetheless, \( \sin \) and \( \cos \) of \( \nu \) are still periodic functions with period \( 1/f_{sat} \). Using the Fourier theorem these signals can be written as sums of sines of \( f_{sat} \) harmonics. The spectra of the \( \nu \) and \( \phi \) functions will become more reach of peaks that result from modulations of \( f_{sat} \) with \( f_{E} \).

**Appendix B**

The influence of \( i \) and \( e \) errors on \( \hat{a} \) can be measured by the derivatives of \( \hat{a} \) with respect to the two orbital parameters. In both cases it is possible to reach relatively simple analytic expressions as follows. Plugging Equation (7) into Equation (20) and solving for \( \hat{a} \) gives

\[ \hat{a} = (Ah)^{\frac{1}{2}}. \]

(32)

where

\[ A = \frac{\sqrt{T}}{2\pi f_{E\text{ef}} + f_{E}} \approx \frac{\sqrt{T}}{2\pi f_{sat}} = a^{\frac{1}{2}} \]

(33)

\[ h = C - C_{n} + x_{d}C_{a}. \]

(34)

Thus, the derivatives of \( \hat{a} \) with respect to \( i \) and \( e \) can be written as

\[ \frac{d\hat{a}}{di} \approx \frac{2}{3} \hat{a} \frac{dh}{di} \]

(35)

\[ \frac{d\hat{a}}{de} \approx \frac{2}{3} \hat{a} \frac{dh}{de} \]

(36)

Next, we make explicit the dependency of \( h \) on \( i \) and \( e \) after recalling that

\[ C_{n} = -C \frac{\cos i}{(1-e^2)^{\frac{3}{2}}} \]

(37)

\[ C_{a} = -C \]

(38)

\[ C_{m} = 1 + C \frac{3 \cos^2 i - 1}{2(1-e^2)^{\frac{3}{2}}} \]

(39)

where

\[ C = \frac{3R^2f_{2}}{2 - a^2}. \]

(40)

Plugging Equations (37), (38) and (39) into Equation (34) we obtain

\[ h = 1 + C \left[ \frac{3 \cos^2 i - 1}{2(1-e^2)^{\frac{3}{2}}} + \frac{\cos i}{(1-e^2)^{\frac{3}{2}}} - x_{d} \right]. \]

(41)

The derivatives of \( h \) with respect to \( i \) and \( e \) are

\[ \frac{dh}{di} = -C \left[ \frac{3 \cos i \sin i}{(1-e^2)^{\frac{3}{2}}} + \sin i + \frac{dx_{d}}{di} \right] \]

(42)

\[ \frac{dh}{de} = -C \left[ \frac{1.5 - 4.5 \cos^2 i}{(1-e^2)^{\frac{3}{2}}} \sin e - \frac{4 \cos i}{(1-e^2)^{\frac{3}{2}}} e + \frac{dx_{d}}{de} \right]. \]

(43)

For small eccentricities, like those we consider in this work, \( (1-e^2) \approx 1 \). Moreover, since \( C \) is small and the term within the square braces in the right-hand side of Equation (41) is never big, then \( h \approx 1 \). Using this approximations and plugging Equations (43) and (42) into Equations (35) and (36) yields

\[ \frac{d\hat{a}}{di} \approx -\frac{2}{3} \hat{a} \frac{\hat{C} \left( 3 \cos i \sin i + \sin i + \frac{dx_{d}}{de} \right)}{3} \]

(44)

\[ \frac{d\hat{a}}{de} \approx -\frac{2}{3} \hat{a} \frac{\hat{C} \left( e (1.5 - 4.5 \cos^2 i - 4 \cos i - 3) + \frac{dx_{d}}{de} \right)}{3}. \]

(45)
The coefficient 1/57 in front of the term on the right-hand side of Equation (44) is introduced to express the sensitivity in m/deg. The derivatives of $x_a$ can be calculated analytically when recalling that $x_a$ is accurately approximated by a polynomial function whose expression is

$$P_3(t, \epsilon) = 0.9 + 0.22t - 2.84t^2 - 1.6t^3 + 1.13t^4 + 0.53t^5 + 0.056t\epsilon - 0.3t^2\epsilon + 0.387t^4\epsilon + 0.032t^5\epsilon - 0.023t^3\epsilon^3 - 0.012t^2\epsilon^4,$$  \hspace{1cm} (46)

where the symbols $t$ and $\epsilon$ denote the coded variables for $i$ and $e$, respectively. In the coding scheme physical intervals $[0 \text{ rad}, 2.09 \text{ rad}]$ and $[0, 0.1]$ of $i$ and $e$ correspond to the same interval $[-1, 1]$ for $t$ and $\epsilon$.

The derivatives in Equations (44) and (45) are strongly dependent on the inclination and to a minor extent on the semi-major axis and eccentricity. The plots in Fig. 13 show exemplary $\frac{d\alpha}{dt}$ and $\frac{d\alpha}{d\epsilon}$ versus inclination plots for orbits with $e = 0.05$ and $a = 7400 \text{ km}$.

In practical applications expecting errors of less than 1 deg for $\dot{i}$ and up to 10% for $\dot{e}$, then the $\dot{\alpha}$ deviations calculated through Equations (45) and (44) never exceed a couple of hundreds of meters for the orbital range considered in this work.

References


